Creating a disposition for critical thinking in the mathematics classroom.

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It has become almost common place to talk about the knowledge explosion and the information economy, but if we take a moment to reflect about what this means we should realise that the advancement in technology and knowledge demands new mathematics to be developed to work at the extremes of research. Not only that – Critical Thinking is one of the key skills in this ever-changing world. In this paper an argument is made for the creation of disruptive spaces in the undergraduate mathematics classroom that should lead to instilling a critical disposition in our students. This is based on research by Facione *et al* that shows that a distinction should be made between the activity of critical thinking and the component skills that fall under it. A characterisation of such possible disruptive spaces is made, drawing on the South African context as well as a brief characterisation of first year engineering mathematics students at the University of the Witwatersrand, as well as the mathematical skills and disposition towards mathematics with which they leave school.

Introduction

How do we educate students to become professional engineers? The Engineering Council of South Africa (ECSA, 2003) gives us ten exit level outcomes. If we look at the exit outcomes as a whole, I believe that the individual we are looking for must fulfil the requirements of what has, mostly in other contexts, been called a Critical Thinker.

It has become almost common place to talk about the knowledge explosion and the information economy, but if we take a moment to reflect about what this means we should realise that the advancement in technology and knowledge demands new mathematics to be developed to work at the extremes of research. Consequently, any profession where mathematics forms any kind of basis to new developments should be demanding of institutions graduates who are willing and able to examine critically any new developments and their applications. "… [Critical Thinking is] one of the key skills for survival in this ever changing world and as such the foundation of an education system should be tailored to this imperative" (Radzi *et al*, 2009).

But can critical thinking skills be taught explicitly in mathematics courses, especially undergraduate mathematics courses? In this paper, when referring to Critical Thinking, a distinction is made between the activity of Critical Thinking, and the component skills associated with it. "Critical thinking is judgement, reflective and purposive" (Facione, 2000), while the component skills referred to are specifically the six skills of Interpretation, Analysis, Evaluation, Inference, Explanation and Self-regulation, as outlined by Peter Facione, "Critical Thinking: What it is and why it counts" (Facione, 2010). The question asked at the beginning of this paragraph and the ideas developed in this paper emerge from personal experience and observation. These are based on ten years of teaching mathematics in South Africa at high school level (old and new curriculum), five years of running the first year engineering mathematics support programme at the University of the Witwatersrand (Wits) and one and a half years of teaching mainstream calculus to first year engineering students at Wits. In addition the author has been a student in both the humanities and mathematics. The observation referred to is that the teaching/learning of "Critical Thinking" falls in the domain of the humanities and does not seem to have had much impact on thinking about mathematics education.

Not that mathematicians or mathematics educators would deny that there are specific thinking skills required in doing and learning mathematics. The skills that one reads about, however, generally fall into mathematics specific skills such as “Procedures”, “Mathematical reasoning” (logical thought, reasoning with mathematical concepts) or “Problem solving”, as opposed to the skills mentioned above, which we would specifically call Critical Thinking skills. The domain of mathematics education is (largely) restricted to mathematics specific skills.

When approaching this question, one would think that giving mathematical content to the six skills outlined by Facione would be the way forward. But, after looking at some of the problems posed to (engineering) first years, it has become clear there is more to becoming a Critical Thinker than learning the skills and practicing to apply them. Although translation between the six critical thinking skills and the names they are given in a mathematical context is very important and a requisite for further study on this subject, it is even more important to examine the disposition to be a critical thinker.

The challenge that will be looked at in this paper lies in inculcating in students the disposition to be critical thinkers. One can learn critical thinking skills, but to be a critical thinker there must be the disposition to question that which is given. This disposition, however, only comes about through constant goading, constantly being required to question the given, through the raising of questions about the origin of the opinion given. A critical disposition is created by providing what I will refer to as “disruptive spaces”, spaces where the acceptance of a commonly accepted opinion is questioned; spaces where the student is almost forced to question. The creation of such a space, however, is not easy.

In the mathematics classroom this generally does not happen – not in the classes I have taught or have attended. It is not part of “the syllabus”. Rather than being taught critical thinking skills explicitly, mathematical skills are learned tacitly. One becomes a skilful problem solver, for example, by doing lots of problems (with some guidance from an expert) and finding and developing your own strategies that generally work – strategies that one cannot necessarily articulate explicitly. Polya’s “How to solve it” (1945) is an example of an attempt to make the expert’s tacit knowledge explicit. And yet even following the four steps in Polya’s programme does not make for a successful problem solver, without the added metacognitive aspect to problem solving (Yong and Kiong, 2005) that a disposition (and mastery of the skills) for critical thinking brings with it. The idea of monitoring or self-regulation immediately comes to mind as an example. In an analysis of the way in which mathematicians go about solving problems, Schoenfeld (2007) shows that a significant portion of time is spent on looking back, checking and evaluating the process at numerous points of the process. All good problem solvers have discovered this strategy, but we rely on our students to discover this in the same way that we did – by solving lots of problems. If they do discover it, they become good at solving mathematical problems. Similarly they will probably discover (or even invent) strategies of their own that differ from those of other good problem solvers.

To a large extent one can ascribe the approach to teaching undergraduates mentioned above, to the nature of the subject. We start with a certain number of axioms and definitions and steadily build on those with theorems and further definitions. The moments that lend themselves to a critical examination of the matter at hand are few and far between. The “disruptive spaces” are minimal, if at all existent.

The mathematics we teach to first year engineers has been well established for more than three hundred years and questioning the basis on which it has been built would simply be counter-productive at this level. Questioning the axioms can of course yield valuable mathematical results as we can observe in the development of spherical and hyperbolic geometry from the questioning of Euclid’s parallel postulate, but this would not really be appropriate at first year level. To try and get students to think, we should be placing them in situations where they have to question the accepted and the given.

The next section is a clarification of the difference between the activity of critical thinking (the disposition to be a critical thinker) and the component skills mentioned before.

Critical thinking skills and the Disposition toward critical thinking

According to the Delphi research project, critical thinking is "... purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as the explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based ..." (Facione, 2000).

Many of the participants in the project, however, maintained that "focusing only on CT skills is not adequate for instructional purposes"; it is equally important to look at the dispositional side (Facione, 2000). Facione and his team developed the California Critical Thinking Disposition Inventory (CCTDI) in order to put research into this area of critical thinking on an empirical basis.

The core critical thinking skills, as mentioned before, are those of Interpretation, Analysis, Evaluation, Inference, Explanation and Self-Regulation. It is difficult to imagine someone who is able to use these skills in approaching a problem and yet is not disposed to go through the arduous process of using them. It is equally difficult to think that a person who is disposed to think critically does not have the required skills. It is exactly to test this apparent contradiction that the CCTDI was developed.

Seven elements of the overall disposition toward critical thinking were identified (Facione et al, 1995). The ideal critical thinker that emerges is inquisitive, systematic, judicious, analytical, thruthseeking, open-minded and confident in reasoning. Looking at the dispositions and the skills themselves it would be easy to think that there would be a strong correlation between each skill and its seemingly correlative disposition. Research done by Facione and his team does not confirm this. Their findings "tend to disconfirm the supposition that there is a one-to-one relationship between each specific CT skill and its supposed correlative disposition" (Facione, 2000).

What does this finding imply for teaching critical thinking? It shows that skill and disposition are separate things in people. "A developmental perspective suggests that skills and dispositions are mutually reinforcing; and hence should be explicitly taught and modelled together" (Kitchener & King in Facione, 2000). Practically, we know that we learn best that which we need to know, in other words, learning follows motivation. "Thus, engendering the desire to use CT as a favoured means of problem solving and decision making prepares the ground for teaching and learning the CT skills" (Facione, 2000).

Up to this point it has been shown that we need to teach critical thinking explicitly as part of an undergraduate mathematics course, that relying on tacit transfer of skills to turn our students into good problem solvers is not enough. But teaching critical thinking skills cannot be done separately from teaching the disposition to think critically – a much harder proposition. Below is a brief characterisation of first year engineering mathematics students encountered over the past five years, to highlight certain issues that compound the problem.

A brief characterisation of first year engineering mathematics students (at Wits over the past five years)

Our students emerge from school with "gaps" in their Procedural knowledge (they can't factorise; can't do long division; can't do simple arithmetic without a calculator; can't follow extended algorithms), Conceptual knowledge (trigonometric functions are a mystery; logs and exponents even worse), Visual/Spatial reasoning that is almost non-existent (they can't represent a situation graphically or interpret graphic representations successfully) and there is a seemingly total inability to tackle "word problems". Clearly the concatenation of the list is a slight exaggeration, although not grossly so. In a baseline test of 15questions administered to 600 first years at the beginning of the year (based on Grade 11 and 12 mathematics), only three questions showed more than a 50% pass rate. Although it is in itself a daunting task to overcome this lack of basic knowledge, both procedural and declarative, this is not the most fundamental stumbling block to their development as critical thinkers and problem solvers.

Students emerge from school with the idea that doing mathematics is memorising a limited number of examples and reproducing this knowledge in a test or exam where similar questions are asked. Every question posed mathematically has a definite and unquestionable answer. The problems that they have been confronted with at school should properly be termed "exercises" – exercises that function only to give them practice in certain skills (Yong & Kiong, 2005). This goes back to a general misconception about mathematics – "mathematics is calculation".

The following statement comes from Malaysia, but it might as well have been written in South Africa, or any number of places in the world:

> The conventional style of lecturing compounded by the heavily packed syllabus taught within constrained time tend to emphasise the learning of facts and solving structurally similar routine problems using fixed mathematical procedures (Radzi *et al*, 2009).

Below is an example of a problem done in a class of around 250 Mechanical and Aeronautical first year engineering students. The aim is to give a clearer understanding of the kinds of problems our students face and also to illustrate the importance of inculcating a disposition to critical thinking amongst students.

Example: Related rates

Question: A car scissors jack is made up of a 4 piece frame each 250mm long, arranged in a rhombus. It is activated by turning a screw thread with a 6mm pitch.

If the handle is turned at a rate of 1 revolution per second, how fast is the car rising when $y = 300$mm? Does it become easier or harder to turn the handle as the car rises?

The question was given as a homework problem to my class of first year Mechanical and Aeronautical engineers, accompanied by a diagram and explanation of what is meant by the pitch of a screw thread.

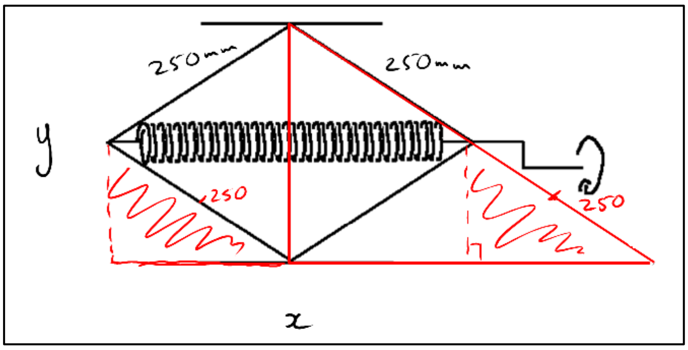

Figure 1. Solution diagram presented in class

This question falls under a section called "Related rates". What is required in such a problem is to relate two rates of change in a given situation, change in *x* and change in *y* in this case. Given what I listed as short-comings in our student body, the difficulties they have to deal with are simply compounded here, just in answering the first part of the question.

The first, and most obvious, obstacle to a student tackling this problem is the language and specifically the interpretation of what is required and what is given. One must know how a scissors jack works, the properties of a rhombus and what the pitch of a screw thread signifies.

Next, the physical object has to be represented geometrically as an abstract form, a rhombus. A successful solution of the problem now depends on reasoning from the properties of this abstract representation (conceptual knowledge). Once the jack is seen as a rhombus, one must realise that the vertical and horizontal lengths can be related by using Pythagoras' theorem. If this is achieved the rest is procedural – finding the rate of change of the vertical with respect to the horizontal requires implicit differentiation. The solution to the problem then depends on the application of some more conceptual knowledge – the chain rule for differentiation:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}.$$

This first part is the part of the question that shows quite clearly the need for the component critical thinking skills mentioned at the beginning of the paper.

The second part of the question I would characterise as an "accidental" disruptive moment. The solution presented in class, using the formula $\text{Power} = \text{Force} \times \text{velocity}$ shows that as the vertical distance increases, the power decreases and, hence, it becomes easier to raise the car as the height increases.

When asked how many of the class had changed a tyre, about 30 of the 250 had done so. Did they find it became easier or harder as the height increased? The majority (all but one) said it became harder. So why does the mathematics say it gets easier?

Included below is an explanation by Dr. M. Bailey-Mcewan from the Wits School of Mechanical Engineering:

> Indeed, the mathematics of the geometry of such a jack do predict that if the jack's handle is being turned at constant rotational speed, and a load of constant mass is being lifted, the power required to do so decreases as the scissors of the jack close, because the vertically upward velocity of their top ends decreases – and, for a constant load represented by a constant force F, the power required is F·v, where v is the [decreasing] velocity. However, this is not the full story. First, the load being lifted is not constant, but increases [rapidly] as the jack lifts the car, because more and more of the car's weight has to be supported. Second, the force required to overcome friction in the jack's screw increases with the increasing weight supported, because the component of this increasing weight along the screw [i.e. the thrust against the screw's threads] increases commensurately. So the frictional force opposing the screw's rotation also increases commensurately. The easiest way to understand this is to recall that a screw is really a spiral, inclined plane. Pushing a block of constant mass up a linear inclined plane requires constant effort, but if the block's mass is increasing, pushing it demands commensurately more effort – the frictional force resisting motion increases!

Clearly one could not reasonably expect such an in-depth understanding of the situation from a group of first year students. However, a short survey was done after doing the question in class. Between 114 and 122 students responded to the questions using their clickers.

One of the questions posed was "Considering most people's observation that it actually becomes more difficult to turn the handle, why do you think there is this disparity between the mathematical model and people's experience?" In response to the question (118 respondents), 16% said that the model was incorrect, 71% said the model was correct, but there are other factors that cause us to feel it gets harder (we get more tired, we need to turn the handle more to achieve the same amount of lift, etc.), 8% said the interpretation of the question was incorrect, while 4% had no opinion on the matter.

When it was pointed out that the answer was not to the question asked, i.e. the answer was that it becomes easier to lift the car, while the question was whether it became easier to turn the handle, 30% said they had noticed this discrepancy, while 70% hadn't (118 responses).

What was concerning about the results was that, although there was clearly some indication that the model should be questioned (the experience of 29 out of 30 students), the majority of students would rather accept the authoritative version to the solution (that offered in class) and try to find other reasons for the discrepancy, than to question the answer.

Changing students' dispositions toward critical thinking in mathematics.

The example above serves to highlight the need for teaching critical thinking explicitly in the mathematics classroom. The majority of students were quite happy to accept an incorrect answer, whether that was because they were simply overwhelmed by the difficulty of the mathematics, or because it was presented by a figure in authority. And this leads to my assertion that it is equally important to inculcate a critical disposition in our students. It is necessary to teach the seven elements of a critical disposition explicitly.

The first step, however, is to get students to become receptive to having their concept of mathematics challenged. To get them to see the importance of thinking critically, it is important to get them out of their complacency about what they believe mathematics to be and what is required to become a good problem solver. It is necessary for the lecturer to become the Socratic gadfly, if you like, stinging them out of their comfort zone by creating space for disruptive moments as part of the lecture or tutorial.

A disruptive moment cannot always be a situation where mathematics is used incorrectly (as in my example), however. Although there may be a place for this, one would rather create such moments in a positive manner. If we look at situations where there is *no* incentive toward critical thinking we may be able to identify characteristics of disruptive moments that *are* desirable.

The practice of solving structurally similar routine problems during lectures and tutorials does not encourage critical and creative thinking (Radzi *et al*, 2009). "In many cases, students in math, science, and engineering learn a sequence of procedures known as algorithms, which they can apply to well-structured, well-defined problem situations" (Dougherty & Fantaske, 1996). Too much emphasis on quantitative problem-solving through "rote-learned algorithms in isolated contexts ... without necessarily understanding underlying concepts ..." means that students go through the course "without developing the capability to draw on their learned experience to deal with new, previously, unseen and, necessarily, more complex situations" (Bowden in Radzi *et al*, 2009).

Here is the first clue, then, to creating a disruptive moment. Exercises (as classified before) are not problems. Exercises have their place in reinforcing certain procedural skills or clarifying the use of certain algorithms, but when we want to create a space where the accepted view of mathematics as calculation and solution of standard problems is challenged, where every problem has a solution that can be found algorithmically, we need to give students the opportunity to think insightfully by "increasing our use of ill-structured problems" (Sternberg & Lubart in Radzi *et al*, 2009). Getting back to the example of the related rates problem, it would mean *not* giving the diagram, *not* explaining what the pitch of the screw means, possibly even leaving that completely out of the question and letting students discover that it is a necessary parameter that they would need to know in order to solve the problem. In other words it is not an incorrect solution that causes the disruption, but the ill-structured nature of the problem. The student is required to give structure to the problem.

An important aspect to problems is that students should be motivated to solve them. Part of the disposition for critical thinking is inquisitiveness and truth-seeking. If the students can see the value of finding a solution they will be more likely to be motivated to find an answer. Years of schooling have "engendered a belief that school mathematics tasks need not make sense" (Goos *et al*, 2000, in Yong & Kiong, 2005). If a task makes no sense, why would you put any kind of meaningful thought into it?

Again, coming back to the example, the (incorrect) solution given to the second part could be presented for debate – from the statistics enough of the students were uncomfortable enough with the solution for meaningful peer discussion to ensue.

The idea above of peer discussion leads to the next clue for creating a meaningful disruptive space. Creating a space where students need to convince each other that they are correct – explain their reasoning and justify their conclusions. This may not sound like something that is easily done in the context of mathematics. And yet, if one follows the methodology of the flipped classroom and concept tests, pioneered by Eric Mazur in the field of physics education, the goal is not as unattainable as it seems.

In experimenting with this idea in lectures as well as tutorials this year, the requisite material was presented as online videos (for the tutorials) and then "concept questions" were posed during face-to-face time. The key for this to work is of course the question. The question should uncover some common misconceptions and present answers corresponding to these, as well as the correct answer, in MCQ format. A good question will give a reasonable spread of answers and students are then required to find someone who disagrees with them and convince them that they are correct. Here there is no authoritative voice giving the correct answer (sometimes all the answers are correct) or a standard version of reasoning through the problem (students can come to the correct answer in different ways). In such a situation the student is encouraged to think carefully for a solution, justify their reasoning and critically evaluate their own arguments as well as those of another, thus encouraging the development of a critical disposition.

What the characteristics of a disruptive space highlight is that it should be carefully planned and framed time-wise. Every moment cannot be disruptive; every exercise cannot be a problem. Given the gaps that our students arrive with and given the pace and level of complexity of the mathematics we need them to master by the end of the first year, one needs to find a balance. Exercises are important to improve student understanding of concepts; routine problems are important to embed facility with certain procedures and algorithms. But without planning and inserting our disruptive spaces carefully and explicitly analysing what comes out of them, we will be failing in our duty to create engineers for the twenty first century.

The role of the lecturer

It should be clear from what has been said above, that the role of the lecturer is of paramount importance, in at least two ways.

The first is that s/he should be clear in what the aim of a problem is. The aim of the example of the related rates question was to give a practical application of related rates. Making clear the practical application of the mathematics we do is part of the motivation for students to think critically about the solution. But essentially we want the students to learn about solving related rates problems. As such, it is more of an exercise than a problem.

If the question is rephrased and put in a manner as I suggested in the previous section, then the aim is to encourage a critical disposition, to awake a sense of curiosity, of seeing the need and experiencing the satisfaction of analysing a problem situation, to encourage evaluation of proposed solutions. So the original aim is actually sacrificed, in a sense, for a higher purpose. Not only should the lecturer be a person who is clear about the goal of a problem, but s/he should be willing to surrender some of the purely mathematical aims of problems in order to achieve wider critical thinking aims.

The second is that the lecturer should be an example of a critical thinker themselves. Such a person should be willing to entertain the notion that there are multiple possible solutions to any given problem situation. But more importantly they should exhibit the dispositions to critical thinking to serve as an example to their students. When one follows a “flipped classroom”/concept test approach, you are in a way giving over your authority to your class, not an easy thing to do. In doing this the lecturer is actually embodying the disposition of open-mindedness. Facione (2000) says that modelling critical thinking is a powerful tool “for nurturing the disposition toward CT in students and co-workers.”

Conclusion

In this paper it has been argued that a critical thinking approach to mathematical problem-solving actually enhances the problem-solver’s abilities and chances of solving a problem. But teaching critical thinking skills without teaching the disposition to think critically is not enough. Skills and dispositions are not the same. They should be taught explicitly and separately.

John Dewey (1933), in “How we think” (as quoted in Facione *et al* 1995) went as far as to say the following about dispositions to critical thinking, which he called personal attributes:

> If we were compelled to make a choice between these personal attributes and knowledge about the principles of logical reasoning together with some degree of technical skill in manipulating special logical processes, we should decide for the former.

The difference between the disposition to think critically and having the skill to do so, is the difference between “what people can do and what they actually do in real-world contexts” (Halpern, 1998).

It is very important, then, to create spaces in the undergraduate mathematics classroom for what I have called disruptive moments. Moments where students are actively encouraged or even stung into critical activity. Such spaces should incorporate the use of ill-structured problems as well as peer-learning. The role of the lecturer is of paramount importance in creating these spaces and choosing problems that will motivate students to engage with them in a critical manner.

> ... [A]ny college which merely trains people for entry-level jobs, yet instils in them no valid general education, no grounding in how to learn, and no disposition to think, does a grave disservice to those graduates and the nation. And, those misguided or impatient students who seek only job training are asking far too little (Facione et al, 1995).

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